

Some Remarks on the Jacobian Conjecture and Drużkowski mappings

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Abstract

In this paper, we first show that the Jacobian Conjecture is true for non-homogeneous power linear mappings under some conditions. Secondly, we prove an equivalent statement about the Jacobian Conjecture in dimension $r \geq 1$ and give some partial results for $r = 2$.

Finally, for a homogeneous power linear Keller map $F = x + H$ of degree $d \geq 2$, we give the inverse polynomial map under the condition that $JH^3 = 0$. We shall show that $\deg(F^{-1}) \leq d^k$ if $k \leq 2$ and $JH^{k+1} = 0$, but also give an example with $d = 2$ and $JH^4 = 0$ such that $\deg(F^{-1}) > d^3$.

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1 Introduction

Throughout this paper, we will write \mathbf{K} for any field with characteristic zero and $\mathbf{K}[x] = \mathbf{K}[x_1, x_2, \dots, x_n]$ for the polynomial algebra over \mathbf{K} with n indeterminates

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$x = x_1, x_2, \dots, x_n$. Let $F = (F_1, F_2, \dots, F_n) : \mathbf{K}^n \rightarrow \mathbf{K}^n$ be a polynomial mapping, that is, $F_i \in \mathbf{K}[x]$ for all $1 \leq i \leq n$. Let $JF = (\frac{\partial F_i}{\partial x_j})_{n \times n}$ be the Jacobian matrix of F . The Jacobian Conjecture (JC) raised by O.H. Keller in 1939 in [Kel] states that a polynomial mapping $F : \mathbf{K}^n \rightarrow \mathbf{K}^n$ is invertible if the Jacobian determinant $\det JF$ is a nonzero constant. This conjecture has been attacked by many people from various research fields, but it is still open, even for $n \geq 2$. Only the case $n = 1$ is obvious. For more information about the wonderful 70-year history, see [BCW], [vdE], and the references therein. It can easily be seen that the JC is true if the JC holds for all polynomial mappings whose Jacobian determinant is 1. We make use of this convention in the present paper.

In 1980, S.S.S.Wang ([Wan]) showed that the JC holds for all polynomial mappings of degree 2 in all dimensions. The most powerful result is the reduction to degree 3, due to H.Bass, E.Connell and D.Wright ([BCW]) in 1982 and A.Yagzhev ([Jag]) in 1980, which asserts that the JC is true if the JC holds for all polynomial mappings of degree 3 (what is more, if the JC holds for all cubic homogeneous polynomial mappings!). It is even shown in [dBvdE2] that the condition that JH is symmetric and H is cubic homogeneous is sufficient. In the same spirit of the above degree reduction method, another efficient way to tackle the JC is the Drużkowski's Theorem ([Dru]): the JC is true if it is true for all Drużkowski mappings (in all dimension ≥ 2). One more interesting result is due to Gorni-Zampieri ([GZ]), who proved in 1997 that there exist Gorni-Zampieri pairings between the cubic homogeneous polynomial mappings and the Drużkowski mappings.

Recall that F is a cubic homogeneous mapping if $F = X + H$ with X the identity (written as a column vector) and each component of H being either zero or cubic homogeneous. A cubic homogeneous mapping $F = X + H$ is a **Drużkowski (or cubic linear) mapping** if each component of H is either zero or a third power of a linear form. Each Drużkowski mapping F is associated to a scalar matrix A such that $F = X + (AX)^{*3}$, where $(AX)^{*3}$ is the **Drużkowski symbol** for the vector $((A_1X)^3, \dots, (A_nX)^3)$ with A_i the i -th row of A . Clearly, a Drużkowski mapping is uniquely determined by this matrix A . In section 2, we prove that the JC is true for Drużkowski mappings in some cases.

Apparently, the notion of a Drużkowski mapping can be easily generalized. Namely, for any positive integer $d \geq 2$, we say that $F = X + H$ is homogeneous power linear of degree d if each component of H is either zero or a d -th power of a linear form. The JC is true in general in case it is true for homogeneous power linear maps of degree d , where d is any integer larger than two. If F is an invertible polynomial map of degree d in dimension n , the degree of its inverse is at most d^{n-1} . This has been proved in [BCW]. But if additionally $F = X + H$ is homogeneous power linear of degree d such that $JH^3 = 0$, then the degree of the inverse of F is at most d^2 . We will prove this in section 5, using results of section

3.

In section 3, we generalize the definition of GZ-paired in [dB1], [vdE] and [GZ]. We use this to prove in section 4 that the Jacobian Conjecture in dimension $r \geq 1$ is equivalent to the Jacobian Conjecture for non-homogeneous power linear maps with $\text{rank} A \leq r$ and prove the Jacobian Conjecture is true in this case for $r = 2$ under the condition that $\det(DJH + I) = 1$, where D is a certain diagonal matrix.

2 The JC for Drużkowski mappings

Theorem 2.1. *Let $F = X + H$ such that $H_i = (A_i X)^{d_i}$ is a power of a linear form for each i , where A_i is the i -th row of a matrix A . If $\text{Tr} JH = 0$ and all the determinants of the $i \times i$ principal minors of A are zero for $2 \leq i \leq n$, then F is a polynomial automorphism.*

Proof. Since $JH(x^{(1)}) + \cdots + JH(x^{(n)}) = DA$, where D is some diagonal matrix and $x^{(i)} \in \mathbf{K}^n$ for $1 \leq i \leq n$, we have that all the determinants of the $i \times i$ principal minors of $JH(x^{(1)}) + \cdots + JH(x^{(n)})$ are zero for $2 \leq i \leq n$. Furthermore, the trace of $JH(x^{(1)}) + \cdots + JH(x^{(n)})$ is zero by additivity. Hence $JH(x^{(1)}) + \cdots + JH(x^{(n)})$ is nilpotent and $\det(JF(x^{(1)}) + \cdots + JF(x^{(n)})) = \det(nI_n + JH(x^{(1)}) + \cdots + JH(x^{(n)})) = n^n$. Thus we deduce from Theorem 3.5 in [GdBDS] that F is invertible. \square

Corollary 2.2. *Let $F = X + H$ be a Drużkowski mapping, say that $H_i = (A_i X)^3$ for each i . If $\det JF = 1$ and all the determinants of the $i \times i$ principal minors of A are zero for $2 \leq i \leq n - 4$, then F is a polynomial automorphism.*

Proof. If there exists an $i \in \{n - 3, n - 2, n - 1, n\}$ such that some $i \times i$ principal minor of A is nonzero, then $\text{corank} A \leq 3$. Therefore, F is a tame automorphism in that case (see Theorem 7.1.1 in [dB1]).

Since $\det JF = 1$, we have $\text{Tr} JH = 0$. If all the determinants of the $i \times i$ principal minors of A are zero for $i \geq n - 3$, then the conclusion follows from Theorem 2.1. \square

Corollary 2.3. *Let $F = X + H$ be a Drużkowski mapping in dimension n . If $\det JF = 1$ and the diagonal of JH is entirely nonzero, then F is tame for $n \leq 9$ and linearly triangularizable for $n \leq 7$.*

Proof. Since $\det JF = 1$ and the diagonal of JH is entirely nonzero, we have $\text{rank} A \leq \lfloor \frac{n}{2} \rfloor$ (see the Theorem in [Yan]). Thus $\text{rank} A \leq 4$ when $n \leq 9$, in which case F is tame (see Theorem 7.1.2 in [dB1]). Furthermore, $\text{rank} A \leq 3$ when $n \leq 7$, in which case F is linearly triangularizable (see Corollary 4.1 in [dBvdE1]). \square

3 Gorni-Zampieri pairing

Definition 3.1. Let $f : \mathbf{K}^r \rightarrow \mathbf{K}^r$ be polynomial maps and $F : \mathbf{K}^n \rightarrow \mathbf{K}^n$ be nonhomogeneous power-linear maps with $n > r$. We say that f and F are GZ-paired (weakly GZ-paired) through the matrices $B \in M_{r,n}(\mathbf{K})$ and $C \in M_{n,r}(\mathbf{K})$ if

- 1) $f(y) = BF(Cy)$ for all $y \in \mathbf{K}^r$,
- 2) $BC = I_r$,
- 3) $\ker B = \ker JH$ ($\ker B \subseteq \ker JH$),

where $H = F - X$.

Lemma 3.2. Let $H = (H_1, H_2, \dots, H_n)^t$ and H_i is a power of $A_i X$ for each i , where A_i is the i -th row of A . Then $\ker JH = \ker A$ and $n - \dim(\ker JH \cap \mathbf{K}^n) = \text{rank} A$.

Proof. Since $JH = \text{diag}\{d_1 t_1^{d_1-1}, \dots, d_n t_n^{d_n-1}\} A$, where $t_i = A_i X$ for $1 \leq i \leq n$, we have $\ker JH = \ker A$ and $n - \dim(\ker JH \cap \mathbf{K}^n) = \text{rank} A$. \square

Theorem 3.3. Let $f : \mathbf{K}^r \rightarrow \mathbf{K}^r$ be a (non)homogeneous polynomial map of degree (at most) d . If $r < n$, then there exists a (non)homogeneous power linear map F of degree d such that f and F are GZ-paired through some matrices $B \in M_{r,n}(\mathbf{K})$ and $C \in M_{n,r}(\mathbf{K})$.

Proof. The proof is similar to the homogeneous case of Theorem 6.2.8 of [dB1], and the cubic homogeneous case in Theorem 6.4.2 of [vdE] and in Theorem 1.3 of [GZ]. \square

Theorem 3.4. Let $F : \mathbf{K}^n \rightarrow \mathbf{K}^n$ be a (non)homogeneous polynomial map of degree d and let $r \geq n - \dim(\ker JH \cap \mathbf{K}^n)$, where $H = F - X$. If $r < n$, then there exist an (non)homogeneous polynomial map f of degree d such that f and F are weakly GZ-paired through some matrices $B \in M_{r,n}(\mathbf{K})$ and $C \in M_{n,r}(\mathbf{K})$.

Proof. Since $\dim(\ker JH \cap \mathbf{K}^n) \geq n - r$, $\ker JH \cap \mathbf{K}^n$ has a linear subspace S of dimension $n - r$. Take B in $M_{r,n}(\mathbf{K})$ such that $\ker B = S$. Then $\text{rank} B = r$ and by $(\ker B \cap \mathbf{K}^n) \subseteq (\ker JH \cap \mathbf{K}^n) \subseteq \ker JH$, we obtain $\ker B \subseteq \ker JH$. Since $\text{rank} B = r$, there exists a C in $M_{r,n}(\mathbf{K})$ such that $BC = I_r$. Now $f := BF(CX)$ has the desired properties. \square

Corollary 3.5. Let $F : \mathbf{K}^n \rightarrow \mathbf{K}^n$ be a (non)homogeneous power linear map of degree d and let $r = \text{rank} A$, where A is defined by $F_i - X_i = (A_i X)^{d_i}$. If F is of Keller type and $d_i \geq 2$ for all i , then $r < n$. If $r < n$, then there exists a (non)homogeneous polynomial map $f : \mathbf{K}^r \rightarrow \mathbf{K}^r$ of degree (at most) d such that f and F are GZ-paired through some matrices $B \in M_{r,n}(\mathbf{K})$ and $C \in M_{n,r}(\mathbf{K})$.

Proof. The claim that $r < n$ under the given conditions follows by looking at the leading homogeneous part of $\det JF \in \mathbf{K}$. So assume that the condition $r < n$ of the second claim is fulfilled.

By lemma 3.2, $r = n - \dim(\ker JH \cap \mathbf{K}^n)$. Hence by the preceding theorem, f and F are weakly GZ-paired through some matrices $B \in M_{r,n}(\mathbf{K})$ and $C \in M_{n,r}(\mathbf{K})$. Since $\ker B \subseteq \ker A$ and $\text{rank} B = r = \text{rank} A$, we have $\ker B = \ker A$. Thus f and F are GZ-paired. \square

Lemma 3.6. *Suppose that f and $F = X + H$ are weakly GZ-paired through matrices $B \in M_{r,n}(\mathbf{K})$ and $C \in M_{n,r}(\mathbf{K})$. Then we have the following.*

i) *If $y \in \mathbf{K}^n$ and $y_0 \in \ker JH$, then $F(y + y_0) = F(y) + y_0$.*

ii) *If $y \in \mathbf{K}^n$, then $CB y - y \in \ker B \subseteq \ker JH$.*

iii) *If $y \in \mathbf{K}^n$, then $H(CB y) = H(y)$.*

Proof. ii) follows from $B(CB y - y) = BCB y - B y = I_r B y - B y = 0$. To prove i), notice first that $F(y + y_0) = F(y) + y_0$ is equivalent to $H(y + y_0) = H(y)$. Next we have

$$H_i(y + y_0) - H_i(y) = \int_0^1 \left(\frac{d}{dt} H_i(y + t y_0) \right) dt$$

and

$$\frac{d}{dt} H_i(y + t y_0) = JH_i(y + t y_0) \cdot y_0 = 0$$

because $y_0 \in \ker JH$. So $H_i(y + y_0) = H_i(y)$ for all i , which gives i). iii) follows by taking $y_0 = CB y - y$ in $H_i(y + y_0) = H_i(y)$ for all i . \square

Remark 3.7. Lemma 3.6 is somewhat similar to Lemma 6.4.4 in [vdE], which is valid with weakly GZ-pairing as well. The purpose of Lemma 3.6 is to replace Lemma 6.4.4 in [vdE] in case not all components of H are powers of a linear form. We need such a replacement in particular for Proposition 6.4.8 iii) in [vdE], which we use later in this article.

4 The JC in dimension r

The following theorem is a special case of (2) \Rightarrow (3) of Theorem 4.2 in [dB2], in which h may be any polynomial.

Theorem 4.1. *Let $F = X + H$ such that $H_i \in \mathbf{K}[h]$ for all i , for a fixed linear form h . If $\det JF = 1$, then F is invertible. More precisely, F is linearly triangularizable.*

Proof. Since $\ker(JH \cap \mathbf{K}^n) \supseteq \ker Jh$ and $\dim \ker Jh = n-1$, we have $\dim \ker(JH \cap \mathbf{K}^n) \geq n-1$. It follows that with F , a polynomial mapping f in dimension $r = 1$ is weakly GZ-paired (see Theorem 3.4). Therefore, F is invertible and F is linearly triangularizable (see Proposition 6.2.7 v) in [dB1]). \square

Remark 4.2. Notice that in the above proof, f and F are GZ-paired, if and only if $\deg H \geq 1$. This is because $\text{rank } JH = 1 = \text{rank } B$, if and only if $\deg H \geq 1$.

Problem 4.3. Let $F = X + H$ and $H_i = (A_i X)^{d_i}$ for $1 \leq i \leq n$. If $\det JF = 1$ and $\text{rank}(A) \leq 2$, then F is invertible.

Theorem 4.4. Problem 4.3 is equivalent to the Jacobian Conjecture in dimension 2.

Proof. We use the invertibility equivalence of GZ-pairing. From Corollary 3.5, we can deduce that F is GZ-paired with a polynomial map in dimension 2 in case $\text{rank}(A) \leq 2$. So if the Jacobian Conjecture is true in dimension 2, then Problem 4.3 has an affirmative answer as well.

On the other hand, if $f : \mathbf{K}^2 \rightarrow \mathbf{K}^2$ is a polynomial map, then f is GZ-paired with a non-homogeneous power-linear map $F_A = X + H$ such that $H_i = (A_i X)^{d_i}$ for each i , with $\text{rank } A \leq 2$ (see Theorem 3.3). So if Problem 4.3 has an affirmative answer, then the Jacobian Conjecture is true in dimension 2. \square

Next, we get a generalized statement of Problem 4.3.

Problem 4.5. Let $F = X + H$ and $H_i = (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^{d_i}$ for $1 \leq i \leq n$. If $\det JF = 1$ and $\text{rank}(A) \leq r$, then F is invertible.

Remark 4.6. We can assume $d_i \geq 2$ for all $1 \leq i \leq n$ in Problems 4.3 and 4.5. This is because we can obtain $d_i \geq 2$ for all $1 \leq i \leq n$ if we replace F by $F(LX) - c$ for a suitable linear map L and a suitable $c \in \mathbf{K}$.

Theorem 4.7. Problem 4.5 is equivalent to the Jacobian Conjecture in dimension r .

Proof. Similar to the proof of Theorem 4.4 \square

Next, we give some partial results about Problem 4.5.

Theorem 4.8. Let $F = X + H$, where $H = (H_1, H_2, \dots, H_n)^t$ and H_i is a homogeneous polynomial of degree d_i for $1 \leq i \leq n$. If $\det(I + DJH|_a) \neq 0$ for each $\lambda \in \mathbf{K} \setminus \{1\}$ and every $a \in \mathbf{K}^n$, where

$$D = \frac{1}{\lambda - 1} \text{diag} \left\{ \frac{1}{d_1}(\lambda^{d_1} - 1), \frac{1}{d_2}(\lambda^{d_2} - 1), \dots, \frac{1}{d_n}(\lambda^{d_n} - 1) \right\}$$

and $JH|_a$ is obtained from JH by replacing X by a , then F is injective on every line that pass through the origin. More precisely, for each $\lambda \in \mathbf{K} \setminus \{1\}$ and every $a \in \mathbf{K}$, we have $(I + DJH|_a)a = 0$, if and only if $F(a) = F(\lambda a)$.

In particular, if $d_1 = d_2 = \cdots = d_n = d \geq 2$, then $\det JF = 1$ is equivalent to $\det(I + DJH) = 1$, so homogeneous Keller maps are injective on lines through the origin.

Proof. Since $H_i = d_i^{-1} \sum_{j=1}^n x_j H_{x_j}$ for $1 \leq i \leq n$, we have $F = (I + D' JH)X$, where $D' = \text{diag}\{d_1^{-1}, d_2^{-1}, \dots, d_n^{-1}\}$. Take $\lambda \in \mathbf{K} \setminus \{1\}$, $a \in \mathbf{K}^n$ and $b = \lambda a$. Then $F(a) = F(b)$ is equivalent to

$$(I + D' JH_a)a = (I + D' JH_b)b,$$

which in turn is equivalent to

$$(I + D' JH_a)a = (\lambda I + D' \Lambda JH_a)a,$$

where $\Lambda = \text{diag}\{\lambda^{d_1}, \lambda^{d_2}, \dots, \lambda^{d_n}\}$. That is,

$$[(\lambda - 1)I + D'(\Lambda - I)JH_a]a = 0.$$

Since $(\lambda - 1)^{-1}D'(\Lambda - I) = D$, we see that $F(a) = F(b)$, if and only if

$$[I + DJH|_a]a = 0,$$

as desired. \square

Remark 4.9. If $d_1 = d_2 = \dots = d_n = d \geq 2$, then Theorem 4.8 is similar to Proposition 1.1 in [YdB].

Lemma 4.10. Suppose that f and F are weakly GZ-paired through matrices B and C . Let $a \in \mathbf{K}^n$ and $\lambda, \mu \in \mathbf{K}$. If $F(\lambda(a + b)) \neq F(\mu(a + b))$ for all $b \in \ker B$, then $f(\lambda Ba) \neq f(\mu Ba)$

Proof. Suppose that $f(\lambda Ba) = f(\mu Ba)$ and $\lambda \neq \mu$. By definition 3.1, we have $BF(\lambda C Ba) = BF(\mu C Ba)$. Hence $F(\lambda C Ba) - F(\mu C Ba) \in \ker B$. By *iii*) (or *i*)) and *ii*) of Lemma 3.6, we have $F(\lambda a) - F(\lambda C Ba) = \lambda(a - C Ba) \in \ker B$ and $F(\mu a) - F(\mu C Ba) = \mu(a - C Ba) \in \ker B$. Consequently, $F(\lambda a) - F(\mu a) \in \ker B$, say that

$$F(\lambda a) - F(\mu a) = (\mu - \lambda)b$$

where $b \in \ker B$. By adding $F(\mu a) + \lambda b$ on both sides, we get $F(\lambda a) + \lambda b = F(\mu a) + \mu b$. Hence we have $F(\lambda(a + b)) = F(\mu(a + b))$ on account of *i*) of Lemma 3.6. This gives the desired result. \square

Theorem 4.11. Let $F = X + H$, where $H = (H_1, H_2, \dots, H_n)^t$ and H_i is a homogeneous polynomial of degree d_i for $1 \leq i \leq n$. If $\ker JH \cap \mathbf{K}^n$ is a space of dimension $\geq n - 2$, and $\det(I + DJH) = 1$ for every $\lambda \in \mathbf{K} \setminus \{1\}$, where

$$D = \frac{1}{\lambda - 1} \text{diag} \left\{ \frac{1}{d_1}(\lambda^{d_1} - 1), \frac{1}{d_2}(\lambda^{d_2} - 1), \dots, \frac{1}{d_n}(\lambda^{d_n} - 1) \right\}$$

then F is invertible.

Proof. Since $\ker JH \cap \mathbf{K}^n$ is a space of dimension $\geq n - 2$, it follows that F is weakly GZ-paired with a polynomial f in dimension 2. Hence

$$\det Jf = \det JF = \det(I + IJH) = \lim_{\lambda \rightarrow 1} \det(I + DJH) = 1.$$

By Theorem 4.8, F is injective on the lines that pass through the origin. Using $\text{rank } B = 2$, Lemma 4.10 subsequently gives that f is injective on the lines that pass through the origin. Thus f is invertible (see [Gwo]). By the invertibility equivalence of GZ-pairing, F is invertible. \square

Corollary 4.12. *Let $F = X + H$, where $H = (H_1, H_2, \dots, H_n)^t$ and $H_i = (a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n)^{d_i}$ for $1 \leq i \leq n$. If $\det(I + DJH) = 1$, where*

$$D = \frac{1}{\lambda - 1} \text{diag} \left\{ \frac{1}{d_1}(\lambda^{d_1} - 1), \frac{1}{d_2}(\lambda^{d_2} - 1), \dots, \frac{1}{d_n}(\lambda^{d_n} - 1) \right\}$$

for any $\lambda \neq 1$, $\lambda \in \mathbf{K}$ and $\text{rank } A \leq 2$, then F is invertible.

Proof. Since $\text{rank } A \leq 2$, the space $\ker JH \cap \mathbf{K}^n$ has dimension $\geq n - 2$. Hence the conclusion follows from Theorem 4.11. \square

5 A bound for the degree of the inverse of some special polynomial maps

We start with a proposition that gives a connection between weak GZ-pairing and invertibility and the degree of the inverse.

Proposition 5.1. *Suppose that f and F are weakly GZ-paired, and one of both is invertible. Then $\deg f^{-1} \leq \deg(F^{-1}) \leq d \cdot \deg f^{-1}$*

Proof. Say that f and F are weakly GZ-paired through matrices $B \in M_{r,n}(\mathbf{K})$ and $C \in M_{n,r}(\mathbf{K})$. If F is invertible, then by $f = BF(Cx)$ and $BC = I_r$, we have $f^{-1} = BF^{-1}(Cx)$. Hence $\deg(f^{-1}) \leq \deg F^{-1}$ if F is invertible.

So assume that f is invertible. By Proposition 6.4.8 iii) in [vdE], we have $F^{-1} = X - H(Cf^{-1}(BX))$, where $H(X) = F(X) - X$. Since $\deg H(X) \leq d$, we have $\deg(F^{-1}) \leq d \cdot \deg f^{-1}$. \square

Write $|_{x=g}$ for substituting $x = x_1, x_2, \dots, x_r$ by $g = g_1, g_2, \dots, g_r$ and $|_{X=G}$ for substituting $X = x_1, x_2, \dots, x_n$ by $G = G_1, G_2, \dots, G_n$. The following theorem is a generalization of Theorem 3(1) in [LDS] to the case where H_i is not a power of a linear form for some i .

Theorem 5.2. *Assume that $f = x + h$ and $F = X + H$ are polynomial maps in dimensions $r < n$ respectively. Then we have the following.*

i) If f and F are weakly GZ-paired, then $Jh^k = 0$ implies $JH^{k+1} = 0$.

ii) If f and F are GZ-paired, then $Jh^k = 0$, if and only if $JH^{k+1} = 0$.

Proof. Assume that f and F are weakly GZ-paired through matrices B and C . Since $x + h = B(Cx + H(Cx))$ and $BC = I_r$, we have $h = BH(Cx)$. Hence $Jh = BJH|_{X=Cx}C$ and $JH|_{X=Cx} \cdot CJh^k B = (JH|_{X=Cx} \cdot CB)^{k+1}$. Substituting $x = BX$ gives

$$(5.1) \quad JH|_{X=CBX} \cdot CJh|_{x=BX}^k B = (JH|_{X=CBX} \cdot CB)^{k+1}$$

Suppose first that $Jh^k = 0$. Then also $Jh|_{x=BX}^k = 0$ and $(JH|_{X=CBX} \cdot CB)^{k+1} = 0$. Hence $J(H(CBX))^{k+1} = 0$. By *iii* of lemma 3.6, we have $H(CBX) = H$, so $J(H(CBX)) = JH$ and $JH^{k+1} = 0$ as well.

Suppose next that f and F are GZ-paired and $JH^{k+1} = 0$. Using $J(H(CBX)) = JH$ and $BC = I_r$, we get $JH|_{X=CBX} \cdot C = JH|_{X=CBX} \cdot CBC = JH \cdot C$. Furthermore, $J(H(CBX)) = JH$ tells us that the right hand side of (5.1) is zero. Hence by $JH \cdot C = JH|_{X=CBX} \cdot C$ and (5.1),

$$JH \cdot CJh|_{x=BX}^k B = JH|_{X=CBX} \cdot CJh|_{x=BX}^k B = (JH|_{X=CBX} \cdot CB)^{k+1} = 0$$

Since $\ker B = \ker JH$, we have $B \cdot CJh|_{x=BX}^k B = 0$. Together with $BC = I_r$, we get $Jh^k = B \cdot CJh|_{x=B(Cx)}^k B \cdot C = 0|_{X=Cx} \cdot C = 0$, as desired. \square

Theorem 5.3. *Let $F = X + H$ be a polynomial map, such that H is homogeneous and $\dim \ker JH = \dim(\ker JH \cap \mathbf{K}^n)$. If $JH^3 = 0$, then F is invertible and $F^{-1} = X - H(X - H)$.*

Proof. Suppose that $JH^3 = 0$. Then we have $r := n - \dim(\ker JH \cap \mathbf{K}^n) = n - \dim \ker JH = \text{rank } JH < n$. Hence by Theorem 3.4, there exists a polynomial map f such that f and F are weakly GZ-paired through some matrices $B \in M_{r,n}(\mathbf{K})$ and $C \in M_{n,r}(\mathbf{K})$. Since $\text{rank } B \leq r \leq \text{rank } JH$ and $\ker B \subseteq \ker JH$, we have $\ker B = \ker JH$. So f and F are GZ-paired through B and C .

Write $h = BH(CX)$. Then h is homogeneous of the same degree d as H is. From Theorem 5.2, $Jh^2 = 0$ follows. We shall show that $Jh \cdot h = 0$. If $d = 0$ then $Jh = 0$. If $d \geq 1$, then by Euler's homogeneous function theorem, $Jh \cdot h = d^{-1}Jh^2x = 0$, where $x = (x_1, x_2, \dots, x_r)^t$. So $Jh \cdot h = 0$ and by Proposition 3.1.2 of [dB1], $2x - f = x - h$ is the inverse polynomial map of f .

Using Prop. 6.4.8 iii) in [vdE], $f^{-1} = 2x - f$, i) of definition 3.1 and $F = X + H$, in that order, we obtain

$$\begin{aligned} F^{-1} &= X - H(Cf^{-1}(BX)) \\ &= X - H(C(2BX - f(BX))) \\ &= X - H(C(2BX - BF(CBX))) \\ &= X - H(CBX - CBH(CBX)) \end{aligned}$$

Now $F^{-1} = X - H(X - H)$ follows by applying *iii*) of Lemma 3.6 twice on the right hand side. \square

Corollary 5.4. *Let $F = X + H$ be a homogeneous power linear map. If $JH^3 = 0$, then $\deg(F^{-1}) \leq (\deg F)^2$*

Proof. By power linearity, $\ker JH = \ker A$ for some $A \in M_n(\mathbf{K})$. Hence the previous theorem gives the desired result. \square

Question 5.5. *Let $F = X + H$ be a homogeneous power linear map. If F is invertible and $JH^{k+1} = 0$, then $\deg(F^{-1}) \leq (\deg F)^k$.*

We see from Corollary 5.4 that Question 5.5 has an affirmative answer for $k \leq 2$. However, it is not true for $k \geq 3$. We will give a counterexample below.

Theorem 5.6. *Suppose that f and F are (weakly) GZ-paired through matrices B and C . Then f and $\tilde{F} := (F, x_{n+1} + (B_i X)^d)$ are (weakly) GZ-paired as well. Furthermore, if f is invertible, then the degree of the last component of the inverse $(F, x_{n+1} + (B_i X)^d)$ is d times the degree of the i -th component of f^{-1} .*

Proof. One can easily see that f and \tilde{F} are (weakly) GZ-paired through matrices \tilde{B} and \tilde{C} , where \tilde{B} is obtained from B by adding a zero column on the right, and \tilde{C} is obtained from C by adding an arbitrary row on the bottom. By Prop. 6.4.8 iii) in [vdE], and by definition of \tilde{B} and \tilde{C} , the last component of $\tilde{F}^{-1}(X)$ equals $x_{n+1} - (B_i C f^{-1}(BX))^d$, which by $BC = I_r$ simplifies to $x_{n+1} - (f^{-1}(BX))_i^d$. The degree of $(f^{-1}(BX))_i$ is equal to that $(f^{-1}(x))_i$, because

$$\deg(f^{-1}(x))_i = \deg(f^{-1}(BCx))_i \leq \deg(f^{-1}(BX))_i \leq \deg(f^{-1}(x))_i$$

which completes the proof. \square

Next we give a counterexample of Question 5.5.

Example 5.7. (Furter) *Let x and h be given by $x = (x_1, x_2, \dots, x_6)^t$ and $h = (2x_2x_6 - 2x_3^2 - x_4x_5, 2x_3x_5 - x_4x_6, x_5x_6, x_5^2, x_6^2, 0)^t$ and $f = x + h$.*

Following Theorem 3.3, we get a homogeneous power linear F with which f is GZ-paired. Suppose f and F are weakly GZ-paired through matrices B and C . Let $n = \dim F$. By Theorem 5.6, we get $\tilde{F} = (F, x_{n+1} + (B_1 X)^2)$ and $\deg(\tilde{F})^{-1} \geq 2 \deg(f^{-1})_1$. Let $\tilde{F} = \tilde{X} + \tilde{H}$. Since $Jh^3 = 0$, we have $J\tilde{H}^4 = 0$ by theorem 5.2. It is easy to compute that $\deg(f^{-1})_1 = 6$. Thus $\deg(\tilde{F}^{-1}) \geq 12 > 8 = 2^{4-1}$.

Remark 5.8. Example 5.7 also shows that the assumption that $\dim \ker JH = \dim(\ker JH \cap \mathbf{K}^n)$ in Theorem 5.3 is necessary.

Notice that $\text{rank } JH \leq k$ implies $JH^{k+1} = 0$ when JH is nilpotent. We get the following question if we replace $JH^{k+1} = 0$ by $\text{rank } JH \leq k$ in Question 5.5.

Question 5.9. *Let $F = X + H$ be a polynomial map over $\mathbf{K}[x_1, x_2, \dots, x_n]$. If F is invertible and $\text{rank } JH \leq k$, then $\deg(F^{-1}) \leq (\deg F)^k$.*

In [YdB] (Theorem 3.4), we showed that Question 5.9 has an affirmative answer in case $\ker JH = \ker JH \cap \mathbf{K}^n$, because $\dim \ker JH = n - \text{rank } JH \geq n - k$. This is in particular the case when F is (non-homogeneous) power linear, see [YdB] (Theorem 3.5).

Theorem 5.10. *If $\text{rank } JH \leq 1$ or $\text{rank } JH \geq n - 1$, then Question 5.9 has an affirmative answer.*

Proof. In [BCW], it has been proved that the degree of the inverse of any invertible polynomial map F is at most $(\deg F)^{n-1}$. This gives the case $\text{rank } JH \geq n - 1$.

So assume that $\text{rank } JH \leq 1$. Reading the proof of Theorem 4.2 in [dB2], we see that there exists a $T \in \text{GL}_n(K)$ such that $T^{-1}H(TX)$ is of the form

$$[c_1, c_2, \dots, c_s, \lambda x_{s+1} + g, h_{s+2}(\lambda x_{s+1} + g), h_{s+3}(\lambda x_{s+1} + g), \dots, h_n(\lambda x_{s+1} + g)]^t,$$

where $0 \leq s \leq n - 1$, $c_i \in \mathbf{K}$ for all i , $\lambda \in \mathbf{K} \setminus \{-1\}$ and $g \in K[x_1, x_2, \dots, x_s]$. One can verify that

$$\left(\frac{1}{\lambda + 1} (\lambda x_{s+1} + \tilde{g}) \right) \Big|_{X=F} = \lambda x_{s+1} + g$$

where $\tilde{g} = g(x_1 - c_1, x_2 - c_2, \dots, x_s - c_s)$, and that the inverse of $T^{-1}F(TX)$ is

$$\left[x_1 - c_1, x_2 - c_2, \dots, x_s - c_s, x_{s+1} - \left(\frac{1}{\lambda + 1} (\lambda x_{s+1} + \tilde{g}) \right), \right. \\ \left. x_{s+2} - h_{s+2} \left(\frac{1}{\lambda + 1} (\lambda x_{s+1} + \tilde{g}) \right), \dots, x_n - h_n \left(\frac{1}{\lambda + 1} (\lambda x_{s+1} + \tilde{g}) \right) \right]^t.$$

Hence F^{-1} has the same degree as F itself. □

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